Instability of an Elastic Circular Plate Subjected to Nonuniform Loads

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The method of eigenfunctions expansion is used for analyzing the stability of a circular plate subjected to nonuniform, normal and shearing, stresses at the circumference. The use of this method together with the Galerkin procedure leads to a relatively simple stability determinant. A numerical example is given for the buckling of a clamped circular plate under two opposite concentrated loads. The results indicate the possibility of buckling in compression and also in tension.

Nomenclature

A_m, B_m, C_m, D_m	= coefficients, Eq. (1)
D	$= Eh^3/12(1-\nu^2)$
I_m	= modified Bessel function
J_n	= Bessel function
L	= operator, Eq. (18)
N_i	= coefficients
P	= concentrated force
R	= radius of the plate
S, T	= eigenvalues
W	= function of ρ
X_{ij}	= defined by Eq. (14a-b)
a_m, b_m, c_m, d_m	= coefficients, Eq. (6)
h	= thickness
i, j, m	= integers
k	= parameter, Eq. (20)
r, θ	= coordinates, see Fig. 1
w	= radial displacement
w_i	= eigenfunction
γ^2	= defined by Eq. (11)
$\delta(i,j)$	= Kronecker delta
$\sigma_r, \sigma_\theta, \tau_{r\theta}$	= stress components
ϕ	= stress function, Eq. (6)
Φ_{ij}	= defined by Eq. (A2)
ρ	$= r/R$
μ	= load concentration parameter

Introduction

THE buckling problem of circular and annular plates subjected to uniform loads at the boundaries has received considerable attention in the literature. The classical studies,[1-3] were followed by extensive investigation on the influence of various boundary conditions,[4-8] orthotropy,[9,10] and variable thickness[11-13] on the buckling load. Recently, the case of an annular plate subjected to unequal uniform radial pressures at both boundaries was also investigated.[14-16]

The majority of the authors discuss the case of uniform loads at the boundaries. One exception is Ref. 17, where a complicated iterative method has been used for the investigation of the buckling of a circular plate under two concentrated forces applied along a diameter. It is recalled that the stability problem of a circular plate subjected to nonuniform loads is of much practical interest, especially in the presence of nonuniform heating.

The present study introduces a simple and direct method for determining the buckling solution of a circular plate under any combination of the applied stresses at the boundary. This is a special case of a more general method,[18] applicable to a wide class of elastic stability problems. The main idea is to describe the buckled shape under nonuniform load, as a linear combination of the possible buckled shapes (eigenfunctions) under uniform load with the same boundary conditions. Then the Galerkin procedure is applied to the equilibrium equations assuring an upper bound solution. The main advantage of this method is its numerical simplicity: because of the orthogonal nature of the chosen eigenfunctions, the eigenvalue appears only along the diagonal of the stability determinant. It is noted that the proposed method resembles the well-established methods of vibrational analysis.

As a numerical example, the stability problem of a clamped circular plate, subjected to opposite normal loads, is solved. Results show the dependence of the critical load on the load concentration parameter. Another interesting result is the possibility of buckling under tension.

Formulation of the Problem

A circular plate is loaded at its circumference by a combination of radial stress $\sigma_r(r=R)$ and shear stress $\tau_{r\theta}(r=R)$, which may be represented as

$$\sigma_r(r=R) = S\left(A_0 + \sum_{m=1}^{\infty} A_m \cos m\theta + \sum_{m=1}^{\infty} B_m \sin m\theta\right) \tag{1a}$$

$$\tau_{r\theta}(r=R) = T\left(C_0 + \sum_{m=1}^{\infty} C_m \cos m\theta + \sum_{m=1}^{\infty} D_m \sin m\theta\right) \tag{1b}$$

where A_m, B_m, C_m, D_m are known coefficients and S, T are unknown numbers (eigenvalues) to be determined later. The requirement of static equilibrium leads to the conditions

$$C_1 = -\frac{S}{T} B_1 \qquad D_1 = \frac{S}{T} A_1 \qquad C_0 = 0 \tag{2}$$

At a certain critical combination of the applied loads (1), the plate will buckle. Let the deflection during buckling be denoted as w. The following equilibrium equation is known to govern the buckled shape,[4]

$$\frac{D}{h}\nabla^4 w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\sigma_r \frac{\partial w}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(\tau_{r\theta}\frac{\partial w}{\partial \theta}\right) + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\tau_{r\theta}\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\sigma_\theta \frac{\partial w}{\partial \theta}\right) \tag{3}$$

where σ_r, σ_θ, $\tau_{r\theta}$ are the prebuckling stresses. Equation (3) has a nontrivial solution only for certain combinations of S and T, which form together the buckling interaction curve S vs T. The problem is to find those curves for various known distributions of the stresses (1) at the boundary.

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Determination of the Prebuckling Stresses

We refer now briefly to the problem of determining the prebuckling stresses. As is well known (e.g., Ref. 19) the stresses can be derived from the Airy stress function ϕ

$$\sigma_r = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad \sigma_\theta = \frac{\partial^2\phi}{\partial r^2} \quad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{4}$$

and ϕ itself is biharmonic

$$\nabla^4\phi = 0 \tag{5}$$

A general solution of Eq. (5) has been given by Mitchell.[19] By considering the boundedness and continuity of the prebuckling displacements for a complete circular plate, a great part of the general solution can be omitted. The remaining part, suitable for our problem, is

$$\phi = b_0r^2 + b_1r^3\cos\theta + d_1r^3\sin\theta + \sum_{m=2}^{\infty}(a_mr^m + b_mr^{m+2})\cos m\theta + \sum_{m=2}^{\infty}(c_mr^m + d_mr^{m+2})\sin m\theta \tag{6}$$

The coefficients a_m, b_m, c_m, d_m are determined by equating the stresses σ_r, $\tau_{r\theta}$, derived from Eqs. (4-6), at the boundary $(r=R)$ to those given by Eq. (1). The result is

$$b_0 = \frac{SA_0}{2} \qquad b_1 = \frac{SA_1}{2R} \qquad d_1 = \frac{SB_1}{2R}$$

$$a_m = -\frac{SmA_m + T(m-2)D_m}{2(m^2-m)R^{m-2}} \qquad b_m = \frac{SA_m + TD_m}{2(m+1)R^m}$$

$$c_m = -\frac{SmB_m - T(m-2)C_m}{2(m^2-m)R^{m-2}} \qquad d_m = \frac{SB_m - TC_m}{2(m+1)R^m} \tag{7}$$

Equations (7) define completely the prebuckling stress distribution. The expressions for the prebuckling stresses (4), written in a nondimensional form, are

$$\sigma_r = S\Big\{A_0 + A_1\rho\cos\theta + B_1\rho\sin\theta + \frac{1}{2}\sum_{m=2}^{\infty}\Big[m\rho^{m-2} - (m-2)\rho^m\Big](A_m\cos m\theta + B_m\sin m\theta)\Big\} + T\Big\{\frac{1}{2}\sum_{m=3}^{\infty}(m-2)\ (\rho^{m-2} - \rho^m)\ (D_m\cos m\theta - C_m\sin m\theta)\Big\} \tag{8a}$$

$$\sigma_\theta = S\Big\{A_0 + 3A_1\rho\cos\theta + 3B_1\rho\sin\theta + \frac{1}{2}\sum_{m=2}^{\infty}[-M\rho^{m-2} + (m+2)\rho^m]\ (A_m\cos m\theta + B_m\sin m\theta)\Big\} + T\Big\{\frac{1}{2}\sum_{m=2}^{\infty}[(m-2)\rho^{m-2} + (m+2)\rho^m]\ (D_m\cos m\theta - C_m\sin m\theta)\Big\} \tag{8b}$$

$$\tau_{r\theta} = S\Big\{A_1\rho\sin\theta - B_1\rho\cos\theta + \frac{1}{2}\sum_{m=2}^{\infty}m(\rho^{m-2} - \rho^m)\ (B_m\cos m\theta - A_m\sin m\theta)\Big\} + T\Big\{\frac{1}{2}\sum_{m=2}^{\infty}[-(m-2)\rho^{m-2} + m\rho^m]\ (C_m\cos m\theta + D_m\sin m\theta)\Big\} \tag{8c}$$

Nature of the Eigenfunctions

We note that the stress function $\phi = -(1/2)sr^2$ $(\sigma_r = \sigma_\theta = -S,\ \tau_{r\theta} = 0)$ represents the case of uniform radial pressure at the boundary. As mentioned previously, this is a well-studied problem. The equilibrium equation (3) becomes

$$(D/h)\nabla^4w + S\nabla^2w = 0 \tag{9}$$

For a complete circular plate the solution of Eq. (9) has the following form

Symmetric case:

$$w^{(SY)} = [K_{1n}J_n(\gamma\rho) + K_{2n}\rho^n]\cos n\theta \quad n=0,\ 1,\ \ldots \tag{10a}$$

Antisymmetric case:

$$w^{(AS)} = [K_{1n}J_n(\gamma\rho) + K_{2n}\rho^n]\sin n\theta \quad n=1,2,\ldots \tag{10b}$$

where

$$\gamma^2 = S(hR^2/D) \tag{11}$$

and K_{1n}, K_{2n} are constants of integration.

Each of the general solutions (10) has to satisfy two boundary conditions, at $\rho=1$, yielding thus a transcendental equation for the eigenvalues. Each of the circumferential waves numbers n is associated with an infinite number of eigenvalues. These eigenvalues are the same for both symmetric (10a) and antisymmetric (10b) modes of buckling, except for $n=0$. Usually only the symmetric case is discussed since it includes also the antisymmetric case shifted by an angle of 90°. But in the present solution we shall need all the buckling modes given by Eqs. (10a-b).

Once the eigenvalues have been determined, one of the constants K_{1n}, K_{2n} can be expressed by the second, determining thus the eigenfunctions (10a-b) up to an arbitrary constant. In what follows, we shall omit this constant, leaving the eigenfunctions completely defined and written in a nondimensional form (i.e., multiplied by a unit length).

Now, we may arrange the eigenfunctions in a certain order. One possible criterion for such an arrangement might be the magnitude of the corresponding eigenvalues. But other and probably better criteria are also possible. In any case, we can write the eigenfunctions in the form:

Symmetric case: $w_i^{(S)} = W_i^{(S)}(\rho)\cos n_i^{(S)}\theta \quad i=1,2,\ldots$ (12a)

Antisymmetric case: $w_i^{(A)} = W_i^{(A)}(\rho)\sin n_i^{(A)}\theta \quad i=1,2,\ldots$ (12b)

It is mentioned again that $W_i^{(S)}$ and $W_i^{(A)}$ are completely defined functions of ρ and each of them depends also on the corresponding eigenvalue γ_i and waves number n_i. Note that in general $\gamma_i^{(S)} \neq \gamma_i^{(A)}$ and $n_i^{(S)} \neq n_i^{(A)}$. The eigenfunctions (12) possess two important features:

The first is the identity (9) valid for both symmetric and antisymmetric cases

$$\nabla^4w_i = -(\gamma_i^2/R^2)\nabla^2w_i \qquad i=1,2,\ldots \tag{13}$$

and the second are the following orthogonality relations

$$\int_0^{2\pi}\int_0^1 w_i^{(S)}\nabla^2w_j^{(S)}\rho d\rho d\theta = \frac{\pi}{R^2}X_{ij}^{(S)}\delta(i,j) \qquad 2i,j=1,2,\ldots \tag{14a}$$

$$\int_0^{2\pi}\int_0^1 w_i^{(A)}\nabla^2w_j^{(A)}\rho d\rho d\theta = \frac{\pi}{R^2}X_{ij}^{(A)}\delta(i,j) \qquad i,j=1,2,\ldots \tag{14b}$$

$$\int_0^{2\pi}\int_0^1 w_i^{(S)}\nabla^2w_j^{(A)}\rho d\rho d\theta = \int_0^{2\pi}\int_0^1 w_i^{(A)}\nabla^2w_j^{(S)}\rho d\rho d\theta = 0 \qquad i,j=1,2,\ldots \tag{14c}$$

where X_{ij} depends on the boundary conditions and $\delta(i,j)$ is the Kronecker delta. Relations (14a-c) are easily proved by considering the total potential of the system.

Eigenfunction Expansion of the Solution

Returning now to the original problem [Eqs. (3) and (8)], we expand the solution in the eigenfunctions (12)

$$w = \sum_{j=1}^{\infty} N_j^{(S)} w_j^{(S)} + \sum_{j=1}^{\infty} N_j^{(A)} w_j^{(A)} \tag{15}$$

where $N_j^{(S)}$ and $N_j^{(A)}$ are constants. The eigenfunctions w_j in Eq. (15) are so determined as to satisfy exactly the same boundary conditions as those of the original problem. As a shortcut, we shall rewrite Eq. (15) in the form

$$w = \sum_{j=1}^{\infty} N_j w_j \tag{16}$$

where the set $\{N_j w_j\}$ includes both $\{N_j^{(S)} w_j^{(S)}\}$ and $\{N_j^{(A)} w_j^{(A)}\}$.

Substituting now Eq. (16) into Eq. (3) and using Eq. (13), we obtain

$$\sum_{j=1}^{\infty} N_j \{\gamma_j^2 \frac{D}{hR^2} \nabla^2 w_j + L[w_j]\} = 0 \tag{17}$$

where γ_j is the known eigenvalue associated with the eigenfunction w_j and L is the operator

$$L[w] = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_r \frac{\partial w}{\partial r}) + \frac{1}{r}\frac{\partial}{\partial r}(\tau_{r\theta}\frac{\partial w}{\partial \theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}(\tau_{r\theta}\frac{\partial w}{\partial r}) + \frac{1}{r^2}\frac{\partial}{\partial \theta}(\sigma_\theta \frac{\partial w}{\partial \theta}) \tag{18}$$

Applying the Galerkin procedure on the equilibrium equation (17) and using the requirement that the determinant of the coefficients should banish for a nontrivial solution, yields the equation

$$\det\left(\int_0^{2\pi}\int_0^1 w_i\{\gamma_j^2 \frac{D}{hR^2}\nabla^2 w_j + L[w_j]\}\rho d\rho d_\theta\right) = 0 \qquad i,j = 1,2... \tag{19}$$

For computing the interaction curve S vs T from Eq. (19), we assume a certain ratio k between S and T, namely

$$T = kS \tag{20}$$

Upon substituting Eq. (20) into Eq. (7), we note, because of Eqs. (8) and (18), that the function Φ_{ij} defined by

$$\Phi_{ij} = \frac{R^2}{\pi S}\int_0^{2\pi}\int_0^1 w_i\, L[w_j]\rho d\rho d\theta \tag{21}$$

is independent of S and T although it depends on the parameter k. This is a symmetric function with respect to the indices i, j (this can easily be shown). Substituting Eq. (21) into Eq. (19) and using Eq. (11) and the orthogonality relation [Eq. (14)] yields the simple equation

$$\det[\Phi_{ij} + (\gamma_j^2/\gamma^2)X_{ij}\delta(i,j)] = 0 \qquad i,j = 1,2,... \tag{22}$$

Equation (22) can be solved by standard numerical methods to obtain the smallest eigenvalue γ^2. Usually, an iterative method based on truncation of the infinite determinant (22) is applied. The order of the truncated determinant is successively increased until suitable convergence to the smallest eigenvalue is achieved. This eigenvalue corresponds to the specific chosen value of the parameter k from Eq. (20). By varying the value of k and repeating the numerical procedure, the interaction curve is constructed.

Thus, the stated problem [Eqs. (1-3)] is reduced to the symmetric equation (22). The main disadvantage of the suggested method is that the natural functions are not always available. On the other hand they are available in many important cases and the eigenfunction expansion can be successfully used.[18,20-22]

If the problem in hand has elastic boundary conditions in which the eigenvalue appears, then the Rayleigh-Ritz procedure replaces the Galerkin procedure, but the method of solution is essentially unchanged.

Example: Buckling of a Clamped Circular Plate Subjected to Opposite Normal Loads

The stresses at the boundary are described by (see Fig. 1)

$$\sigma_r(r=R) = -Se^{\mu(\cos 2\theta - 1)} \tag{23a}$$

$$\tau_{r\theta}(r=R) = 0 \tag{23b}$$

where μ is the load concentration factor. For $\mu = 0$, Eq. (23a) is reduced to the uniform case. As μ increases Eq.(23a) approaches a concentrated force. The nondimensional force P is defined by

$$\pi\frac{D}{R}P = -h\int_{-\pi/2}^{+\pi/2}\sigma_r(r=R)Rd\theta \tag{24}$$

or after substituting Eq. (23a) and integrating

$$P = \frac{1}{\pi}\gamma^2 e^{-\mu} I_0(\mu) \tag{25}$$

The Fourier representation of Eq. (23a) is

$$\sigma_r(r=R) = -Se^{-\mu}[I_0(\mu) + 2\sum_{m=2,4,...}^{\infty} I_{m/2}(\mu)\cos m\theta] \tag{26}$$

The prebuckling stresses obtained from Eqs. (1), (7), (8), and (26) are

$$\sigma_r = -Se^{-\mu}\left\{I_0(\mu) + \sum_{m=2,4,...}^{\infty} I_{m/2}(\mu)[m\rho^{m-2} - (m-2)\rho^m]\cos m\theta\right\} \tag{27a}$$

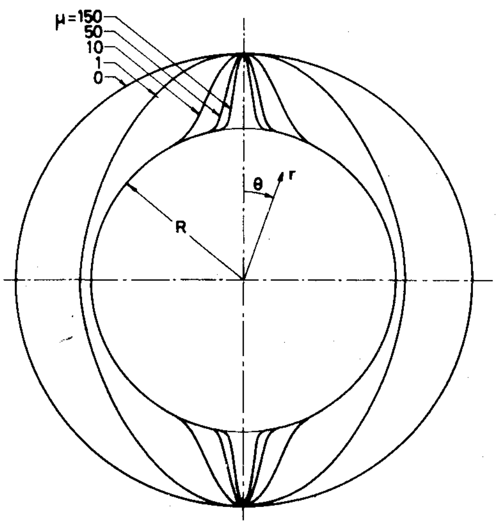

Fig. 1 Stress distribution at the boundary.

$$\sigma_\theta = -Se^{-\mu}\Big\{I_0(\mu) + \sum_{m=2,4,\ldots}^{\infty} I_{m/2}(\mu)\,[-m\rho^{m-2} + (m+2)\rho^m]\cos m\theta\Big\} \tag{27b}$$

$$\tau_{r\theta} = Se^{-\mu}\sum_{m=2,4,\ldots}^{\infty} I_{m/2}(\mu)\,m(\rho^{m-2}-\rho^m)\sin m\theta \tag{27c}$$

For simplicity, a clamped boundary will be considered

$$\rho = 1 \qquad w = 0 \qquad \partial w/\partial\rho = 0 \tag{28}$$

Substituting Eq. (10) into Eq. (28) yields the known equation for the eigenvalues under uniform load

$$J_{n+1}(\gamma) = 0 \quad n = 0,1,\ldots \text{ in the symmetric case}$$
$$n = 1,2,\ldots \text{ in the antisymmetric case} \tag{29}$$

Roots of Eq. (29) are tabulated in Ref. 23 and are used in the subsequent numerical analysis. The eigenfunctions are given by Eq. (12) with

$$W_i(\rho) = \rho^{n_i} - \frac{J_{n_i}(\gamma_i\rho)}{J_{n_i}(\gamma_i)} \qquad i = 1,2,\ldots \tag{30}$$

for symmetric and antisymmetric buckling. However, γ_i and n_i have different values for each mode. An important question arises here concerning the method of ordering the eigenfunctions. An answer to this question is suggested later.

Considering now the stability determinant (22), we obtain from Eq. (14a-b) after substituting Eqs. (12) and (30) and integrating

$$X_{ij} = -\tfrac{1}{2}\gamma_i^2 \tag{31}$$

for both modes. This simple result is obtained by using King's integrals for Bessel functions.[24]

The expression for Φ_{ij} from Eq. (21) is more complicated and is given in the Appendix. Substituting Eq. (31) into Eq. (22) yields the final form

$$\det\left(\frac{2\Phi_{ij}}{(\gamma_i\gamma_j)^2} - \frac{1}{\gamma^2}\delta(i,j)\right) = 0 \quad i,j = 1,2,\ldots \tag{32}$$

where Φ_{ij} is given by Eq. (A2).

Numerical Results

The two main steps of the numerical procedure are the evaluation of Φ_{ij} from Eq. (A2) and the solution for the eigenvalues from Eq. (32).

It is easy to verify that there are four basic modes of buckling; thus the stability determinant (32) separates into four independent determinants according to symmetry and antisymmetry of the modes with respect to the lines $\theta = 0\ldots\pi$ and $\theta = -(\pi/2)\ldots(\pi/2)$. The eigenfunctions for each of these modes follow from Eq. (12)

SS mode: $w_i = W_i\cos n_i\theta \qquad n_i = 0,2,4,\ldots$ (33a)

AS mode: $w_i = W_i\cos n_i\theta \qquad n_i = 1,3,5,\ldots$ (33b)

SA mode: $w_i = W_i\sin n_i\theta \qquad n_i = 2,4,6,\ldots$ (33c)

AA mode: $w_i = W_i\sin n_i\theta \qquad n_i = 1,3,5,\ldots$ (33d)

and W_i is given by Eq. (30).

Each of the waves numbers n_i is associated with an infinite number of eigenvalues that result from Eq. (29). It has been found, after some numerical experiments, that the best arrangement of the eigenfunctions, from the viewpoint of rapid convergence, is in increasing order of the wave numbers where the eigenfunction that corresponds to the lowest eigenvalue of each n_i is taken. The first 10 eigenvalues and wave numbers are shown in Table 1. Note that although n_i and γ_i are equal for the AS and AA modes, the eigenfunctions differ by the trigonometric term (33b,d).

The smallest eigenvalues were computed from Eq. (32) through a standard numerical procedure. Similarly, the evaluation of ϕ_{ij} from Eq. (A2) has been done with existing standard programs. The elements of Eq. (A2) as well as the integrations of Eqs. (A5-A7) have been computed until convergence to six significant figures has been reached. The order of the determinant (32) was increased until convergence of γ^2 to four significant figures for the smallest positive eigenvalue and to three significant figures for the smallest negative eigenvalue was achieved. The highest order used was 25.

The results are shown in Table 2, where the nondimensional critical force (25) is given for various values of the load concentration parameter and for the four modes of buckling. Positive values of P correspond to compression and negative values to tension.

It is seen that as μ increases, an asymptotic value is approached by P. This value, denoted by P_{cr}, corresponds to the cases of instability under opposite concentrated forces. In compression, it is achieved in the SS mode with

$$P_{cr} = 11.92 \tag{34}$$

whereas in tension the SA mode dominates

$$P_{cr} = -364 \tag{35}$$

It is recalled that the result obtained by Rozsa in Ref. 17 for a plate with a free boundary, compressed by two opposite forces, is

$$P_{cr} = 1.071 \tag{36}$$

Finally, we can use the above results in order to give an approximate expression for the eigenvalues γ^2: as μ increases

Table 1 Values of wave numbers n_i and eigenvalues γ_i [read: $(n_i)\,\gamma_i$]

i	SS mode	AS mode	SA mode	AA mode
1	(0) 3.83170	(1) 5.13562	(2) 6.83170	(1) 5.13562
2	(2) 6.38016	(3) 7.58834	(4) 8.77148	(3) 7.58834
3	(4) 8.77148	(5) 9.93610	. . .	. . .
4	(6) 11.08637	(7) 12.22509	. . .	. . .
5	(8) 13.35430	(9) 14.47550	. . .	. . .
6	(10) 15.58984	(11) 16.69824	see case SS	see case AS
7	(12) 17.80143	(13) 18.89999		
8	(14) 19.99443	(15) 21.08514		
9	(16) 22.17149	(17) 23.25677		
10	(18) 24.33825	(19) 25.41714		

Table 2 Critical force $P=(1/\pi)\gamma^2 e^{-\mu} I_0(\mu)$

	SS		AS		SA		AA	
μ	Compression	Tension	Compression	Tension	Compression	Tension	Compression	Tension
0	14.68		26.38		40.71		26.38	
1	14.17		18.74		40.33		43.03	
10	12.40	−565	14.00	−619	36.79	−564	39.94	−616
25	12.14	−432	13.77	−468	35.64	−429	37.93	−468
50	12.04	−400	13.69	−429	35.17	−395	37.19	−431
75	12.00	−391	13.66	−416	35.02	−384	36.96	−420
100	11.98	−387	13.65	−410	34.95	−379	36.85	−415
250	11.94	−380	13.63	−398	34.84	−369	36.71	−407
500	11.92	−378	13.62	−394	34.82	−366	36.68	−404
1000	11.92	−377	13.62	−392	34.81	−364	36.67	−403

one has

$$e^{-\mu} I_0(\mu) \approx 1/\sqrt{2\pi\mu} \tag{37}$$

Hence, for large μ

$$\gamma^2 \approx \pi\sqrt{2\pi\mu} P_{cr} \tag{38}$$

Conclusions

The method of eigenfunctions expansion has been used for solving the buckling problem of a circular plate subjected to nonuniform loads. The use of this method together with the Galerkin procedure leads to a simple stability determinant that can be solved by standard numerical methods.

As an example, the buckling of a clamped circular plate subjected to opposite normal loads is solved. Results indicate the possibility of buckling in tension as well as in compression.

Appendix: Expression for Φ_{ij}

Φ_{ij} is defined in Eqs. (21) and (18). Integration by parts and using the boundary conditions (28) yields the symmetric result

$$\Phi_{ij} = -\frac{1}{\pi S}\int_0^{2\pi}\int_0^1 \left[\rho\sigma_r \frac{\partial w_i}{\partial \rho}\frac{\partial w_j}{\partial \rho} + \tau_{r\theta}\left(\frac{\partial w_i}{\partial \rho}\frac{\partial w_j}{\partial \theta} + \frac{\partial w_i}{\partial \theta}\frac{\partial w_j}{\partial \rho}\right) + \frac{1}{\rho}\sigma_\theta \frac{\partial w_i}{\partial \theta}\frac{\partial w_j}{\partial \theta}\right] \mathrm{d}\rho\, \mathrm{d}\theta \tag{A1}$$

Substituting the prebuckling stresses from Eq. (27) and the eigenfunctions from Eqs. (12) and (30) into (A1), integrating over θ and arranging, yields

$$\begin{aligned}\Phi_{ij} &= \tfrac{1}{2}\gamma_i\gamma_j e^{-\mu} I_0(\mu)\,\delta(i,j) \\ &+ \{\pm\} e^{-\mu} \sum_{m=2,4,\ldots} I_{m/2}(\mu) A_{mij}\delta(m,n_i+n_j) \\ &+ e^{-\mu} \sum_{m=2,4,\ldots} I_{m/2}(\mu) B_{mij}\delta(m,|n_i-n_j|) \\ &- e^{-\mu} \sum_{m=2,4,\ldots} I_{m/2}(\mu) C_{mij}\delta(m,n_i-n_j)\end{aligned} \tag{A2}$$

where the plus sign in the second term appears in the symmetric case and the minus sign in the antisymmetric case. It is also noted that

$$\delta(m,n_i-n_j) = -\delta(m,n_j-n_i) \tag{A3}$$

The coefficients A_{mij}, B_{mij}, and C_{mij} are given by

$$A_{mij} = K_{mij} + \tfrac{1}{2}m(K_{m-2,ij} - K_{mij} + M_{mij} + M_{mji}) \tag{A4}$$

$$B_{mij} = L_{mij} + \tfrac{1}{2}m(L_{m-2,ij} - L_{mij}) \tag{A5}$$

$$C_{mij} = \tfrac{1}{2}m(M_{mij} - M_{mji}) \tag{A6}$$

and (the prime denotes differentiation with respect to ρ)

$$\left\{\begin{matrix} K_{mij} \\ L_{mij}\end{matrix}\right\} = \int_0^1 \rho^{m+1} W_i' W_j' \mathrm{d}\rho + \left\{\begin{matrix} - \\ + \end{matrix}\right\} n_i n_j \int_0^1 \rho^{m-1} W_i W_j \mathrm{d}\rho \tag{A7}$$

$$M_{mij} = n_j \int_0^1 (\rho^{m-2} - \rho^m) W_i' W_j \mathrm{d}\rho \tag{A8}$$

where W_i is given by Eq. (30).

Addendum

The present problem has also been considered by Yamaki,[25] using the Rayleigh-Ritz procedure, with a limited number of terms. For a plate with simply supported boundaries, subjected to the compression of two concentrated forces, Yamaki obtained

$$P_{cr} = 4.182$$

whereas for a clamped plate his result is

$$P_{cr} = 12.8$$

which is higher than the present result, Eq. (34), by about 7.5%.

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